Normal forms of parametrizations of curves and distance between curves

Van Duc Hoang
XLIM UMR 7252 Université de Limoges CNRS
van-duc.hoang@unilim.fr

In this work, we describe we show an mathematical framework to manipulate and do optimization on a space of “paths” or “trajectories” over an affine space $E$. For us a “path” or a “trajectory” is the curve defined as the image of a suited parametrization from $[0, 1]$ to $E$. The set of parametrization considered is the set $\text{Emb}([0, 1], E)$ of differentiable embeddings from $[0, 1]$ to $E$. We denote $\text{Diff}([0, 1])^+$ the group of increasing diffeomorphism from $[0, 1]$ to $[0, 1]$. Two parametrizations $g$ and $\delta \in \text{Emb}([0, 1], E)$ define the same curve if and only if there exists $\varphi \in \text{Diff}([0, 1])^+$ such that $\gamma = \delta \circ \varphi$. The set of curves we want to consider is “identified” to $\mathcal{C} = \text{Emb}([0, 1], E) / \text{Diff}([0, 1])^+$. This approach can be extend to closed curves (taking $S^1$ instead of $[0, 1]$) and our approach can be compared to [Younes08] and [Michor07] but here we do not consider the left action of a transformation group of $E$.

If $\alpha \in \text{Emb}([0, 1], E)$, we denote $l_\alpha$ its length and $L_\alpha: [0, l_\alpha] \to E$ its arc length parametrization. We define by $\mathcal{N}_\alpha ; t \in [0, 1] \mapsto L_\alpha(t) \in E$. We show that the fact that $\alpha \in \text{Emb}([0, 1], E)$ implies that $\mathcal{N}_\alpha \in \text{Emb}([0, 1], E)$ and that $\mathcal{N}_\alpha \circ \varphi = \mathcal{N}_\alpha$ in a way that $\mathcal{N}$ provides a normal forms system for the action of $\text{Diff}([0, 1])^+$ over $\text{Emb}([0, 1], E)$. We identify $\mathcal{C}$ with $\{\mathcal{N}_\alpha | \alpha \in \text{Emb}([0, 1], E)\}$.

With this representation of $\mathcal{C}$, we show that if $C = \gamma([0, 1])$ and $D = \delta([0, 1])$ then $d(C, D) = \int_0^1 ||\mathcal{N}_\gamma(t) - \mathcal{N}_\delta(t)||\, dt$ is a distance on $\mathcal{C}$ (it is to say that it does not depend on the choice of the parametrizations of $C$ and $D$). We then describe the manifold structure of $\mathcal{C}$ : we study the quotient topology and the one induce by $d$ and we show that the action of $\text{Diff}([0, 1])^+$ goes to the tangent space of $\text{Emb}([0, 1], E)$ and that the projection map $\mathcal{N}$ is a differentiable map giving $\mathcal{C}$ and differential manifold structure.

Joint work with: Olivier Ruatta (XLIM UMR 7252 Université de Limoges CNRS)

Bibliography
