Hybrid linearized proximal alternating minimization for solving discrete Mumford-Shah model

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The Mumford-Shah (MS) model [1] is one of the most influential in image segmentation and restoration since it is able to perform jointly image restoration and contour detection. In this work, we propose an adaptation of the Proximal Alternating Linearized Minimization (PA(L)M) algorithm, with proved convergence, to the minimization of a discrete counterpart of the MS model, based on the Ambrosio-Tortorelli (AT) functional [2].

Generalized Discrete MS – Following discretization ideas proposed in [1], a discrete counterpart of the MS functional can be written:

$$\underset{\mathbf{u},\mathbf{e}}{\text{minimize}}\ \mathcal{L}(\mathbf{u},\mathbf{z}) + \beta\|(1-\mathbf{e})\odot D\mathbf{u}\|^2 + \lambda\mathcal{R}(\mathbf{e}), \tag{1}$$

where $\mathbf{z}\in\mathbb{R}^N$ denotes the grayscale, possibly degraded, input image, $\mathbf{u}\in\mathbb{R}^N$ the image to recover, $\mathbf{e}\in\mathbb{R}^M$ is equal to 1 when a contour change is detected and 0 otherwise, β,λ are positive parameters, $D\in\mathbb{R}^{M\times N}$ models a finite difference operator, $\mathcal{L}$ is a data fidelity term and $\mathcal{R}$ denotes a penalization term that favors sparse solution, which is a discrete translation of "short contour".

A new proximal algorithm – One of the most popular choice is $\mathcal{R}_{AT}(\mathbf{e}) = \varepsilon\|D\mathbf{e}\|_2^2 + \frac{1}{4\varepsilon}\|\mathbf{e}\|_2^2$, with $\varepsilon\to 0$ [2]. However, numerically, it is not possible for ε to be arbitrarily small since it controls the thickness of the contours. Hence, we propose a new quadratic-ℓ_1 penalization, to both reproduce the quadratic behavior of $\frac{1}{4\varepsilon}\|.\|_2^2$ for small ε and enforce sparsity, defined as follows:

$$(\forall\mathbf{e}=(\mathbf{e}_i)_{1\le i\le M}\in\mathbb{R}^M)\quad \mathcal{R}_{1,Q}(\mathbf{e}) = \Sigma_{i=1}^M \max\left\{|\mathbf{e}_i|, \frac{\mathbf{e}_i^2}{4\varepsilon}\right\}. \tag{2}$$

This new formulation offers the possibility to deal with different degradation models (e.g. Gaussian or Poisson noise, blur), and with a large panel of regularization terms. We propose a new algorithmic scheme, called Hybrid Linearized Proximal Alternating Minimization (HL-PAM) aiming to combine one step of PAM with one step of PALM allowing to relax condition of a stepsize parameter. The convergence proof is derived.

Contributions – We propose to revisit AT functional by replacing the smooth penalization over $\mathbf{e}$ with nonsmooth penalizations such as ℓ_0, ℓ_1, $\ell_{1,2}$, quadratic-ℓ_1. Such nonsmooth penalizations require the design of a new nonconvex algorithmic scheme, that we called HL-PAM, for which convergence guarantees are derived. Numerical experiments show that the proposed method is able to detect sharp contours and to reconstruct piecewise smooth approximations with low computational cost. We also compare the results with state-of-the-art relaxations of the MS functional and a recent discrete formulation of the Ambrosio-Tortorelli functional.

Fig. 1: Image denoising with quadratic-ℓ_1 penalization. Left to right: data g, approximation u, contours v.

Joint work with: Nelly Pustelnik, Laurent Condat.

References

[1] D. Mumford and J. Shah. Optimal approximations by piecewise smooth functions and associated variational problems. *Comm. Pure Applied Math.*, vol. 42, no. 5, pp. 577-685, 1989.

[2] L. Ambrosio and V.M. Tortorelli. On the approximation of free discontinuity problems. *Boll. Un. Mat. Ital.*, vol. 6, no. B, pp. 105-123, 1992.

[3] M. Foare, N. Pustelnik, and L. Condat. A new proximal method for joint image restoration and edge detection with the Mumford-Shah model. in *Proc. Int. Conf. Acoust. Speech Signal Process.*, (ICASSP 2018), Calgary, Canada.