A low-rank approach to off-the-grid sparse deconvolution

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Introduction We aim to recover accurately the amplitudes $a_i \in \mathbb{C}$ and positions $x_i \in \mathbb{T}^d$ ($\mathbb{T}$ is the torus) of a discrete Radon measure $\mu_0 = \sum_{i=1}^r a_i \delta_{x_i}$ given low-resolution and noisy observations $y = \Phi\mu_0 + w$. The measurements consist in a finite number of Fourier moments $\Phi\mu = \int_{\mathbb{T}^d} \phi(x) \mathrm{d}\mu(x)$, with $\phi(x) = (c_k e^{-2i\pi\langle k,\, x\rangle})_{k \in [\![-f_c, f_c]\!]^d}$. Such super-resolution problems naturally arise in medical, astronomical or microscopy imaging. They can be tackled using total variation regularization, which generalizes ℓ^1-regularization to measure spaces [2]:

$$\min_{\mu \in \mathcal{M}(\mathbb{T}^d)} \frac{1}{2\lambda} \|y - \Phi\mu\|^2 + |\mu|(\mathbb{T}^d), \tag{1}$$

where $|\mu|(\mathbb{T}^d)$ is the total variation of μ, *i.e.* its total mass, and $\lambda > 0$ depends on the noise level $\|w\|$. Our main contribution is a new solver for this infinite-dimensional problem, which requires only $O(f_c^d \log f_c)$ elementary computations per iterations, thus making it scalable in multi-dimensional settings ($d > 1$).

Semidefinite relaxations In 1-D, (1) may be solved exactly by lifting to a semidefinite program in $O(f_c^{2d})$ variables [1, Section 4]. In 2-D and beyond, it may be approximated to arbitrary precision by semidefinite liftings of increasing size [3]. While usual interior points methods are limited in these high-dimensional settings, the solver we propose scales well with the size of the problems. The semidefinite relaxation of (1) reads

$$\min_{\tau, z, u} \left\{ \tau + u_0 + \frac{1}{2} \| \frac{y}{\lambda} + 2z\|^2 \;;\; (a) \left| \begin{array}{l} \mathcal{R} = \begin{bmatrix} R & \tilde{z} \\ \tilde{z}^h & \tau \end{bmatrix} \succeq 0, \\ \tilde{z}_k = z_k, \quad k \in [\![-f_c, f_c]\!]^d \end{array} \right. \text{ and } \quad (b)\, R = \sum_{k \in]\!]-m, m[\![^d} u_k \Theta_k \right\}, \tag{2}$$

with $\Theta_k = \theta_{k_d} \otimes \ldots \otimes \theta_{k_1}$, where θ_{k_j} is the (Tœplitz) matrix with ones on the k_j-th diagonal and zeros elsewhere, and $\otimes$ is the Kronecker product. R may be interpreted as a moment matrix associated to the measure μ_λ solving (1); in particular, one may retrieve from R the positions and amplitudes of the spikes composing μ_λ.

Proposition 1 (Low-rank solutions) *In 1-D, (2) admits a solution $\mathcal{R}_\lambda$ such that* $\operatorname{rank} \mathcal{R}_\lambda \leq r$, *$r$ being the number of Diracs composing μ_λ. This result appears to hold in 2-D (from numerical evidences).*

FFT-based Frank-Wolfe solver To solve (2), we penalize constraint (b) in the objective f and apply Frank-Wolfe algorithm [4] to the resulting semidefinite program. Iterates are stored as $\mathcal{R} = \mathcal{U}\mathcal{U}^*$. Frank-Wolfe's oracle over the semidefinite cone is given by a leading eigenvector of ∇f, which we compute using power iterations. This is done efficiently in $O(f_c^d \log f_c)$, exploiting the connection between Tœplitz matrices and the Fast Fourier Transform. We further add a non convex BFGS update on $\mathcal{U}$ after each Frank-Wolfe step. Our algorithm appears to converge in exactly r steps, r being the number of spikes in μ_λ. We will soon have results on data from the Single-Molecule Localization Microscopy (SMLM) challenge [5].

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References

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