Approximation of multivariate functions with anisotropic mixed smoothness properties – sharp constants and preasymptotics –

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A practically highly relevant model assumption for multivariate functions is based on a bounded mixed derivative. The most classical space of functions with bounded mixed derivative is the Sobolev space $H^r_{mix}$ with the smoothness parameter $r$ being a natural number. The space consists of $L_2$-functions $f$ such that the mixed weak derivative $Df = \partial_{x_1}^r \cdots \partial_{x_d}^r f$ is bounded in $L_2$. In this talk we study the approximation of $d$-variate periodic functions from a related tensor product Sobolev space

$$H^r_{mix} = H^{r_1} \otimes \cdots \otimes H^{r_d},$$

where we have in general different fractional smoothness parameters $r_i$ in every direction. It is known since the 1960s that the asymptotic rate of convergence of the approximation numbers (singular numbers) $a_n$ of the embedding in $L_2$ is determined by the smallest smoothness parameter and the number of its occurrence. In special cases we even observe a sharp dimension-free rate of convergence such as $n^{-r_1}$ if, for instance, $r_1$ is smaller than all remaining smoothness parameters $r_i$. Hence, the underlying dimension $d$ does not seem to affect the order of approximation in this special case. However, so far nothing is said about the $d$-dependence of the constants behind. As a first result we precisely determine the precise behavior of the constants in the most general case and observe an exponential dependence in $d$. Consequently, the mentioned asymptotic error bounds get useless if $n$ is “small”, say $n \leq (1 + \gamma)^d$. This range is called the “preasymptotic range” and represents the only relevant range for computational issues if $d$ is large. As a second main result we characterize the behavior of the approximation numbers in this preasymptotic range by providing a new combinatorial approach towards the number of grid points in anisotropic hyperbolic crosses.

**Joint work with:** Thomas Kühn (Leipzig), Winfried Sickel (Jena)

**References**
