Approximation of multivariate functions with anisotropic mixed smoothness properties – sharp constants and preasymptotics –

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A practically highly relevant model assumption for multivariate functions is based on a *bounded mixed derivative*. The most classical space of functions with bounded mixed derivative is the Sobolev space H^r_{mix} with the smoothness parameter r being a natural number. The space consists of L_2-functions f such that the mixed weak derivative $Df = \partial^r_{x_1} \cdots \partial^r_{x_d} f$ is bounded in L_2. In this talk we study the approximation of $d-$variate periodic functions from a related tensor product Sobolev space

$$H^{\mathbf{r}}_{mix} = H^{r_1} \otimes \cdots \otimes H^{r_d} \,,$$

where we have in general different fractional smoothness parameters r_i in every direction. It is known since the 1960s that the asymptotic rate of convergence of the approximation numbers (singular numbers) a_n of the embedding in L_2 is determined by the smallest smoothness parameter and the number of its occurrence. In special cases we even observe a sharp dimension-free rate of convergence such as n^{-r_1} if, for instance, r_1 is smaller than all remaining smoothness parameters r_i. Hence, the underlying dimension d does not seem to affect the order of approximation in this special case. However, so far nothing is said about the $d-$dependence of the constants behind. As a first result we precisely determine the precise behavior of the constants in the most general case and observe an exponential dependence in d. Consequently, the mentioned asymptotic error bounds get useless if n is "small", say $n \leq (1+\gamma)^d$. This range is called the "preasymptotic range" and represents the only relevant range for computational issues if d is large. As a second main result we characterize the behavior of the approximation numbers in this preasymptotic range by providing a new combinatorial approach towards the number of grid points in anisotropic hyperbolic crosses.

Joint work with: Thomas Kühn (Leipzig), Winfried Sickel (Jena)

References

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