# Kinematic interpretation of quaternionic–Bézier curves and surfaces


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Quaternion–Bézier (QB) surfaces were introduced in [2] as a tool for parametrizing Dupin cyclides and more general Darboux cyclides. Recently in [4] it was noticed that Darboux cyclides are orbits of simple 2-parameter motions in $\mathbb{R}^3$. This was an initial insight that inspired the results presented in the current abstract.

Let us define a rational motion of several parameters as a motion with only rational trajectories. The group of rigid body displacements $SE(3)$ can be identified with the Study quadric $\mathbb{S} \subset \mathbb{R}P^7$ minus certain exceptional 3-dimensional projective subspace $E$: using the dual quaternions $h = p+\varepsilon q$, $p, q \in \mathbb{H}$, $\varepsilon^2 = 0$ as the homogeneous coordinates in $\mathbb{R}P^7$, the equations of $\mathbb{S}$ and $E$ are $p\bar{q} + q\bar{p} = 0$ and $p = 0$, respectively.

A dual quaternion $h = p + \varepsilon q \in \mathbb{S} \setminus E$ acts on $\mathbb{R}^3 = \mathrm{Im}\mathbb{H}$ by Study's kinematic mapping

$$x \mapsto \frac{px\bar{p} + p\bar{q} - q\bar{p}}{p\bar{p}} = \frac{px\bar{p} - 2q\bar{p}}{p\bar{p}} = (px - 2q)p^{-1}. \tag{1}$$

If $h = p + \varepsilon q$ is polynomial of several variables $t_1, \ldots, t_n$ with dual quaternion coefficients then for any fixed $x \in \mathbb{R}^3$ the formula (1) defines the fraction of quaternion polynomials $(px - 2q)p^{-1}$, i.e. for $n = 1, 2$ this is a QB curve or surface, if the Bernstein basis is used. Therefore, motion polynomials of 1 and 2 variables define QB curves and surfaces as their trajectories.

This kinematic interpretation of QB curves and surfaces leads to several conclusions:

- According to [3] for a given rational trajectory curve there is a constructive procedure how to find the corresponding rational motion of minimal degree; the same algorithm can be applied directly for QB representation of any rational curve.
- Any rational surface patch on the sphere can be uniquely represented as QB-surface of twice less degree (see [1, 2]); in the kinematic interpretation this gives the unique spherical motion of minimal degree generalizing [3].
- The latter result cannot be extended to arbitrary rational 2-parameter motions; a counter-example will be presented: the translational surface generated by two non-cospherical circles in $\mathbb{R}^3$.

Other results are related to rational parametrizations of Darboux cyclides, which can be of three topological types (here singular cases are excluded): (a) torus topology $T^2$; (b) one real spherical component $S^2$; (c) two real spherical components $S^2 \sqcup S^2$. In the case (a) several different bilinear QB parametrization were reported in [2]. Here we present multi-linear parametrizations in $\mathbb{S}$ that covers several bilinear cases at once. The case (b) corresponds to oval 2-dimensional quadratics in $\mathbb{S}$ which generate triangular QB-patches of degree 2. The case (c) is the most complicated, since $\mathbb{R}$-birational parametrization is not possible, and both components are separately parametrized by certain quartic surfaces in $\mathbb{S}$ which define QB-patches of bidgeree $(1, 2)$.

## References


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