The nonlocal p-Laplacian Evolution Problem on Random Graphs : The Continuum Limit

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We study numerical approximations of the nonlocal p-Laplacian evolution problem with homogeneous Neumann boundary conditions on inhomogeneous random convergent graph sequences. More precisely, we deal with the problem

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) = -\boldsymbol{\Delta}_p^K(u(x,t)), & x \in \Omega, t > 0, \\ u(x,0) = g(x), & x \in \Omega, \end{cases} \tag{$\mathcal{P}$}$$

where $p \in]1, +\infty[$, $\Omega = [0,1]$, the kernel $K(\cdot,\cdot)$ is a symmetric measurable mapping in $L^\infty(\Omega)$ and

$$\boldsymbol{\Delta}_p^K = -\int_\Omega K(x,y)\big|u(y,t) - u(x,t)\big|^{p-2}(u(y,t) - u(x,t))dy.$$

Problem ($\mathcal{P}$) has various applications to model diffusion phenomena, in particular in signal and image processing. In practice, such an evolution equation is implemented in discrete form (in space and time) as a numerical approximation to the continuous problem, where the kernel is replaced by the adjacency matrix of a graph. To carry out this task, practically one computes the solution of the following discrete problem using a partition (not necessarily uniform) $\{\tau_h\}_{h=1}^N$, $N \in \mathbb{N}^*$ of the time interval $[0,T]$, i.e; $\tau_{h-1} := \big|t_h - t_{h-1}\big|$ and $u_i^h := u(x_i, t_h)$

$$\begin{cases} \dfrac{u_i^h - u_i^{h-1}}{\tau_{h-1}} = \dfrac{1}{n} \displaystyle\sum_{j:(i,j)\in E(G_n)} \big|u_j^{h-1} - u_i^{h-1}\big|^{p-2}(u_j^{h-1} - u_i^{h-1}), \\ u_i^0 = g_i, i \in \{1, \cdots, n\}. \end{cases} \tag{$\mathcal{P}_{n,\tau}^d$}$$

The graph $G_n = (V(G_n), E(G_n))$ is defined by the set of vertices $V(G_n) := \{1, \cdots, n\}$ and the set of edges $E(G_n)$. Thus, ($\mathcal{P}_{n,\tau}^d$) induces a discrete diffusion process parametrized by the structure of the graph G_n. The natural question that arises is to understand the structure of solutions to the discrete problem ($\mathcal{P}_{n,\tau}^d$) and study their continuous limit. Indeed, many real world graphs are random and highly inhomogenous [1], that's why we focus here on these graph models (as we have done the same analysis for deterministic graphs in [2]). Relying on the graph limits theory for random inhomogenous graphs combined with sharp concentration inequalities, we prove convergence of solutions of discrete problems to the solution of the continuous problem with high probability. Moreover, we deliver the corresponding convergence rate with overwhelming probability and show the influence of the choice of p. More precisely, we deal first with a random network model generated by a deterministic sequence of nodes. Based on the error bound obtained for this particular case, we treat secondly the totally random model, we prove convergence of solutions for the discrete model to the solution of the continuous problem as the number of vertices grows. To get the corresponding convergence rate, a supplementary assumption is added regarding the kernel K and the initial data g, that is belonging to appropriate Lipschitz spaces which contain in particular BV functions. This allows us to identify different asymptotic regimes ($n \to +\infty$) depending on the values of p and the smoothness parameters of the Lipschitz spaces.

Joint work with: Jalal Fadili and Abderrahim Elmoataz.

References

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