Approximation of parametric functions with colliding jumps

G. Welper
University of Southern California
welper@usc.edu

One central task in the simulation of parametric and stochastic PDEs is the approximation of the solutions $u(x, \xi)$ that depend on physical variables involved in the PDE and additional deterministic or random parameters ξ. A common approach is to solve the PDE with some fixed parameters giving *snapshots* $u(\cdot, \xi_i)$, $i = 1, \dots, n$ and then to "interpolate" the solution at the unknown parameter values ξ, e.g. by reduced basis methods, empirical interpolation or stochastic collocation. This is fairly well understood for elliptic and parabolic problems, but becomes difficult if the function u has jumps in parameter as one frequently finds in hyperbolic PDEs, level-set methods or elliptic problems with parameter dependent jumps in the diffusion.

In [1], a new interpolation method has been introduced that aligns the jump discontinuities across several snapshots via a transform of the physical domain before interpolation. This way the discontinuities become "invisible" to the interpolation process, which therefore can achieve high convergence rates. Since the jumps typically appear on curves (in 2d) and surfaces (in 3d) this alignment has similarity to shape transformations in imaging. But there are also some differences, e.g. the quality of the alignment is only implicitly assessed via the interpolation error.

The matching becomes challenging when the topology of the jump sets change with parameter. In that case a simple transform as proposed in [1] is no longer possible. Since these topology changes are localized in parameter space, one can treat them by localizing the interpolation of the transformed snapshots accordingly, similar to *hp* finite element methods [2]. Of course simple local refinements are prone the curse of dimensionality and therefore not expected to work well in high parameter dimensions. We therefore provide a tensor-like construction as a starting point for high dimensional interpolation schemes along the lines of sparse grids or other sparse tensor methods.

An alternative approach is based on the observation that the transforms in [1] are defined via ODEs, which take physical variables as initial values and parameters as dynamic variables. From this perspective, the interpolation of the transformed snapshots then picks interpolation points along the trajectory of these ODEs. At topology changes of the jump set these trajectories intersect, so that the ODEs themselves are necessarily singular. We can exploit this observation by providing an algorithm that efficiently predicts the location of these singularities via local eigenvalue problems and then uses interpolation points along the trajectories that do not pass these singularities, comparable to WENO interpolation for hyperbolic problems. This approach requires the solution of several challenges, among others the fact that the functions defining the ODEs have sharp gradients, are only seen implicitly via the interpolation error and must be trained from scarce available data.

References

[1] G. Welper, Interpolation of Functions with Parameter Dependent Jumps by Transformed Snapshots, *SIAM Journal on Scientific Computing*, https://arxiv.org/abs/1505.01227 39(4):A1225-A1250, 2017.

[2] G. Welper *h* and *hp*-adaptive Interpolation by Transformed Snapshots for Parametric and Stochastic Hyperbolic PDEs *Preprint*, https://arxiv.org/abs/1710.11481.