A varifold approach to surface approximation and curvature estimation on point clouds

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We propose a natural framework for the study of surfaces and their different discretizations based on varifolds. Varifolds have been introduced by Almgren in 1965 ([1]) to carry out the study of minimal surfaces. Though mainly used in the context of rectifiable sets, they turn out to be well suited to the study of discrete type objects as well. Let us briefly explain what a d–varifold is: it is a Radon measure on $\Omega \times G_{d,n}$ where $G_{d,n} = \{d\text{–vector plane of } \mathbb{R}^n\}$ is the d–Grassmanian. It can be equivalently understood as the data of a Radon measure μ on $\mathbb{R}^n$ and a probability measure ν_x on $G_{d,n}$ at each point x in the support of μ. Using this point of view, we can easily associate a d–varifold with a d–submanifold M of $\mathbb{R}^n$: we take the surface measure for μ (the d–Hausdorff measure restricted to M, which can be weighted) and for ν_x, we take the Dirac mass at the tangent plane T_xM on $G_{d,n}$. Loosely speaking, mass and tangent planes are enough to define a varifold. Hence, given a finite set of points $\{x_i\}_{i=1\ldots N} \subset \mathbb{R}^n$, weighted by masses $\{m_i\}_{i=1\ldots N} \subset \mathbb{R}_+$, and provided with directions $\{P_i\}_{i=1\ldots N} \subset G_{d,n}$, we associate the d–varifold

$$V_N = \sum_{i=1}^{N} m_i \, \delta_{(x_i,P_i)} \, .$$

While the structure of varifold is flexible enough to adapt to both regular and discrete objects, it allows to define variational notions of mean curvature and second fundamental form based on the divergence theorem (see [2], [3]). Thanks to a regularization of these weak formulations, we propose a notion of discrete curvature (actually a family of discrete curvatures associated with a regularization scale) relying only on the varifold structure. We prove nice convergence properties involving a natural growth assumption: the scale of regularization must be large with respect to the acuracy of the discretization. We performed numerical computations of mean curvature and Gaussian curvature on point clouds in $\mathbb{R}^3$ to illustrate this approach.

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References

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