Triangulating stratified manifolds I: a reach comparison theorem

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We define the reach for submanifolds of Riemannian manifolds, in a way that is similar to the Euclidean case, see [2]. Given a d-dimensional submanifold $\mathcal{S}$ of a smooth Riemannian manifold $\mathcal{M}$ and a point $p \in \mathcal{M}$ that is not too far from $\mathcal{S}$ we want to give bounds on local feature size (or local reach in the sense of [2], see also [1]) of $\exp_p^{-1}(\mathcal{S})$. Here $\exp_p^{-1}$ is the inverse exponential map, a canonical map from the manifold to the tangent space. Bounds on the local feature size of $\exp_p^{-1}(\mathcal{S})$ can be reduced to giving bounds on the reach of $\exp_p^{-1}(\mathcal{B})$, where $\mathcal{B}$ is a geodesic ball, centred at c with radius equal to the reach of $\mathcal{S}$. Equivalently we can give bounds on the reach of $\exp_p^{-1} \circ \exp_c(\mathbf{B}_c)$, where now $\mathbf{B}_c$ is a ball in the tangent space $T_c\mathcal{M}$, with the same radius. To establish bounds on the reach of $\exp_p^{-1} \circ \exp_c(\mathbf{B}_c)$ we use bounds on the metric and on its derivative in Riemann normal coordinates. These bounds are based on the Toponogov comparison theorem, see [4], and an extension due to Kaul [5].

This result is a first step towards answering the important question of how to triangulate stratified manifolds. Triangulating manifolds with boundary, submanifolds of Riemannian manifolds and most generally stratified manifolds, is an important problem because of applications in dynamical systems, physics, and chemistry. Examples of stratified manifolds in applications include conformation spaces of molecules, such as discovered for cyclo-octane [6], and also the invariant sets that appear in dynamical systems [3].

Joint work with: Jean-Daniel Boissonnat.

References

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