Sparse non-negative super-resolution – simplified and stabilised

Bogdan Toader
University of Oxford
toader@maths.ox.ac.uk

We consider the problem of localising point sources on an interval from possibly noisy measurements. More specifically, consider a *non-negative* Borel measure $x = \sum_{i=1}^{k} a_i \delta_{t_i}$ supported on the interval $I = [0,1]$ with $a_i > 0, \forall i$, a real-valued, continuous function ϕ such as the Gaussian $\phi(t) = e^{-\frac{t^2}{\sigma^2}}$ and the noisy measurements $\{y_j\}_{j=1}^m$ taken at locations $\{s_j\}_{j=1}^m$:

$$y_j = \int_I \phi(t - s_j)x(dt) + \eta_j = \sum_{i=1}^{k} a_i \phi(t_i - s_j) + \eta_j, \tag{1}$$

where η_j is the additive noise with $\|\eta\|_2 \le \delta$. We then consider the following *feasibility* problem:

$$Find \quad \hat{x} \ge 0 \quad subject \quad to \quad \sum_{j=1}^{m} \left| y_j - \int_I \phi(t - s_j)\hat{x}(dt) \right|^2 \le \delta^2, \tag{2}$$

over all non-negative Borel measures $\hat{x}$ supported on I. We study how solutios $\hat{x}$ of (2) compare with the measure x from (1) both in the noise-free and the noisy setting.

Noise-free results

In the absence of noise ($\delta = 0$), we show that the true measure x is the only solution to (2) provided that the number of measurements m satisfies $m \ge 2k + 1$ and the measurement functions $\{\phi(t - s_j)\}_{j=1}^m$ form a Chebyshev system[2], namely a system of continuous functions that behave like algebraic polynomials. There is no need here for minimum separation between sources. These results build on and substantially simplify the prior work by Schiebinger et al in [1], which relies on the total variation norm of $\hat{x}$ as a sparsifying penalty.

Samples with additive noise

For $\delta > 0$, we show that the feasibility program is robust against noise, with the error gradually decreasing as the noise level decreases, namely

$$d_{GW}(x, \hat{x}) \le F \cdot \delta, \tag{3}$$

where d_{GW} is the generalised Wasserstein distance between the true measure x and a solution $\hat{x}$ to (2) and F depends on the parameters of the problem (the locations of $\{t_i\}_{i=1}^k, \{s_j\}_{j=1}^m$ and the measurement function ϕ) and on a specific construction of the dual certificate $q(t) = \sum_{j=1}^m b_j \phi(t - s_j)$, but not on δ.

Furthermore, for ϕ Gaussian and minimum separation between sources $\Delta = \min_{i \ne j} |t_i - t_j|$, we give an explicit bound on F which depends on the number of point sources k, the width of the Gaussian σ, and Δ. This is achieved by imposing further conditions on the locations of the samples s_j. We require to have two samples close to each point source t_i, which allows us construct the dual certificate $q(t)$ and control the magnitudes of its coefficients $\{b_j\}_{j=1}^m$.

These results contribute to the growing literature of super-resolution, see for example Candès 2012, Tang 2013, Bendory 2015, Peyré 2015, Fernandez-Granda 2016.

Joint work with: Armin Eftekhari, Jared Tanner, Andrew Thompson, Hemant Tyagi.

References

[1] G. Schiebinger, E. Robeva, B. Recht. Superresolution without separation. *Information and Inference: a journal of the IMA*, 00:1–30, 2017.

[2] S. Karlin, W.J. Studden. Tchebycheff systems: with applications in analysis and statistics. Interscience Publishers, 1966.