

A convex approach to the (Gilbert-) Steiner problem

Mauro Bonafini

Department of Mathematics, University of Trento

`mauro.bonafini@unitn.it`

The Steiner Tree Problem (STP) can be described as follows: given N points $P_1, \dots, P_N$ in a metric space X find a connected graph $F \subset X$ containing the points P_i and having minimal length. Such an optimal graph F turns out to be a tree and is thus called a *Steiner Minimal Tree* (SMT). In case $X = \mathbb{R}^d$, $d \geq 2$, endowed with the Euclidean ℓ^2 metric, one refers often to the *Euclidean* or *geometric* STP.

The situation in the Euclidean case is theoretically well understood: given N points $P_i \in \mathbb{R}^d$ a SMT connecting them always exists, the solution being in general not unique (e.g. for symmetric configurations of the endpoints P_i). The SMT is a union of segments connecting the endpoints, possibly meeting at 120° in at most $N - 2$ further branch points, called *Steiner points*.

The question has been widely studied through combinatoric optimization techniques and finding a Steiner Minimal Tree is known to be a NP hard problem (and even NP complete in certain cases). Nonetheless, the problem has recently attracted a lot of attention in the Calculus of Variations community and several authors have proposed different approximations of it, mainly in the planar case and using a phase field based approach together with some coercive regularization.

In this talk, moving from the formulation of [2], we propose a convex approach for STP and for the Gilbert-Steiner irrigation problem [3] in arbitrary dimension. As first step we reformulate these problems in terms of an optimization over a suitable family of tensor valued measures and then a convex relaxation of the resulting energy is proposed by means of a local convex envelope of the relevant functional. The formulation we obtain this way maintains its validity in any dimension and, with the appropriate modifications, also in manifolds. Furthermore we can prove the sharpness of this formulation whenever minimizers can be calibrated in the sense of [2].

From a numerical point of view we end up with a conic problem involving a set of $\approx 2^{N-1}$ quadratic constraints which can then be solved either via a conic solver or via primal-dual proximal schemes. A localization procedure is also implemented so to focus the computational effort only around the optimal 1d structure. Extensive examples in $\mathbb{R}^2$, $\mathbb{R}^3$ and in the surface case are presented.

Joint work with: Giandomenico Orlandi (University of Verona), Édouard Oudet (University of Grenoble).

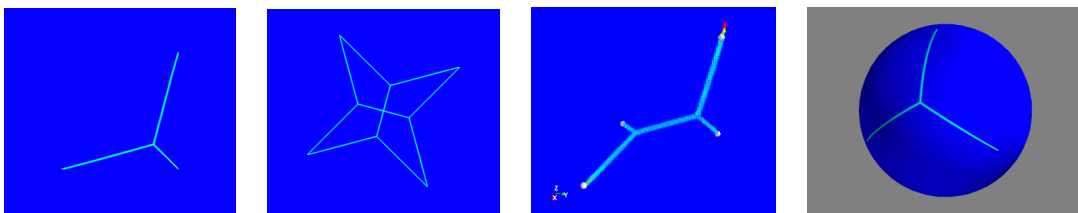


Figure 1: Minimizers of the convex formulation for 3 and 4 points in $\mathbb{R}^2$, 4 points in $\mathbb{R}^3$ and 3 points on a sphere.

References

- [1] Mauro Bonafini, Giandomenico Orlandi, and Édouard Oudet. Variational approximation of functionals defined on 1-dimensional connected sets: the planar case. preprint arXiv:1610.03839v3, 2016.
- [2] Andrea Marchese and Annalisa Massaccesi. The Steiner tree problem revisited through rectifiable G -currents. Adv. Calc. Var., 9(1):1939, 2016.
- [3] Qinglan Xia. Optimal paths related to transport problems. Commun. Contemp. Math., 5(2):251279, 2003.