

Non-Asymptotic behaviour of the spectrum of the Sinc Kernel Operator

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The aim of this paper is to give precise non-asymptotic estimates to the eigenvalues associated to the prolate spheroidal wave functions (PSFWs). PSFWs are the eigenvectors $(\psi_{n,c})_{n \geq 0}$ of the time and band limiting operator given by the sinc-kernel $\mathcal{Q}_c f(x) = \int_{-1}^1 \frac{\sin c(y-x)}{\pi(y-x)} f(y) dy$. They are the $[-c, c]$ -band limited functions that are the most concentrated in the time-interval $[-1, 1]$ and have been introduced into signal processing in the work of Landau, Pollak and Slepian [4, 5].

The corresponding eigenvalues $(\lambda_n(c))_n$ play important roles in a wide range of scientific areas such as signal processing, mathematical-physics, random matrices, numerical analysis including spectral methods, etc. Nowadays, there exists a rich literature devoted to the asymptotic behavior as well as the numerical computation of the eigenvalues $(\lambda_n(c))_n$, see [1] and references therein. These works show that the eigenvalues exhibit three kinds of behavior:

- for $n \ll n_c := \frac{2}{\pi}c$ where $\lambda_n(c) \approx 1$
- for $n \gg n_c$ where $\lambda_n(c) \rightarrow 0$ at super-exponential speed
- a plunge region $n \approx n_c$ of width $\approx \log c$ of transition between the two behaviors.

The aim of this talk is to present *non-asymptotic* estimates. Our results are the following:

Theorem. Let $c > 0$ then,

- for $0 \leq n < \frac{c}{\sqrt{2} \log c}$, $1 - \frac{4}{\sqrt{c}} \frac{(2c)^n}{n!} e^{-c} \leq \lambda_n(c) < 1$;
- for every $\delta \geq 1$, there exists η_δ such that, for $\frac{2c}{\pi} + \delta(\log c + 9) \leq n \leq c$,

$$\lambda_n(c) \leq \frac{1}{2} \exp \left(-\eta_\delta \frac{n - 2 - \frac{2c}{\pi}}{\log c + 5} \right);$$

- for $n \geq \max(2, ec/2)$, $\lambda_n(c) \leq \exp \left(-(2n+1) \log \frac{2}{ec} (n+1) \right)$.

This completes earlier non-asymptotic estimates by A. Israel [3]. Applications to spacing of eigenvalues of random matrices and to the degree of freedom of the *sinc*-kernel operator will be given.

Joint work with: Aline Bonami, Abderrazek Karoui

References

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