Delaunay triangulations of regular hyperbolic surfaces

Iordan Iordanov
LORIA, INRIA Nancy - Grand Est, Université de Lorraine, CNRS UMR 7503, France
iordan.iordanov@loria.fr

Delaunay triangulations and their dual objects, Voronoi diagrams, have been widely studied and have many uses in a vast array of disciplines. In the Euclidean plane, the Delaunay triangulation of a set of points P is the partition of the convex hull of P into triangles whose circumscribing disk does not contain any point of P in its interior. This definition was extended to other settings, including closed Euclidean manifolds [2]. We study Delaunay triangulations of a closed orientable hyperbolic surface $\mathbb{M}_g$ of genus g. The surface $\mathbb{M}_g$ is defined as the quotient of the hyperbolic plane under the action of the group Γ generated by the hyperbolic translations that identify opposite sides of the regular $4g$-gon F_g centered at the origin.

The hyperbolic case is more challenging than the Euclidean case, since hyperbolic translations do not commute. For a finite set of points P in the hyperbolic plane, the set ΓP of images of its elements under the action of Γ is infinite, and, therefore, also its Delaunay triangulation; if the projection of this triangulation onto the surface $\mathbb{M}_g$ is a simplicial complex, then it is the Delaunay triangulation of $\mathbb{M}_g$ defined by the points in P [1].

A sufficient *validity condition* so that this projection is a simplicial complex is that $\delta_P < \frac{1}{2}\,\mathrm{sys}(\mathbb{M}_g)$, where δ_P is the diameter of the largest disks not containing any point of ΓP, and $\mathrm{sys}(\mathbb{M}_g)$ is the systole of $\mathbb{M}_g$, i.e., the minimal length of a non-contractible loop on $\mathbb{M}_g$ [1]. Based on this condition, we generalize the algorithm for closed Euclidean manifolds to the case of the hyperbolic surface $\mathbb{M}_g$: *(a) construct a triangulation of* $\mathbb{M}_g$ *defined by a set of "dummy" points* Q_g *that satisfy the validity condition; (b) insert the points of* P *into the triangulation; (c) remove from the triangulation every point in* Q_g *whose removal does not violate the validity condition.*

We introduce a data structure, relying on the following *inclusion property*: for a set of points P satisfying the validity condition, a face of the Delaunay triangulation of ΓP with at least one vertex inside F_g is contained in the union of F_g and its adjacent images under the action of Γ. This property allows us to implement the algorithm. We present several different strategies for the generation of sets of dummy points Q_g that satisfy the validity condition. We describe the construction of the Delaunay triangulation of $\mathbb{M}_g$ defined by Q_g, as well as the details of subsequent point insertion and removal. We discuss the algebraic expressions arising from the geometric predicates used by the algorithm, and we report experiments.

This work generalizes previous work tailored to the Bolza surface $\mathbb{M}_2$, the most symmetric hyperbolic surface of genus 2, for which the Dirichlet region of the origin is a regular octagon [1, 4].

Joint work with: Matthijs Ebbens, Monique Teillaud and Gert Vegter.

This talk is complementary to the talk of M. Ebbens [3], who treats the mathematical aspects of the algorithm. Among others, he provides an explicit expression for $\mathrm{sys}(\mathbb{M}_g)$, presents a proof for the inclusion property, and performs an analysis of the cardinality of the dummy point sets Q_g in terms of the genus g.

References

[1] M. Bogdanov, M. Teillaud and G. Vegter. Delaunay triangulations on orientable surfaces of low genus. In *Proceedings of the Thirty-second International Symposium on Computational Geometry*, pages 20:1–20:15, 2016. doi:10.4230/LIPIcs.SoCG.2016.20

[2] Manuel Caroli and Monique Teillaud. Delaunay triangulations of closed Euclidean d-orbifolds. *Discrete & Computational Geometry*, 55(4):827–853, 2016. url:https://hal.inria.fr/hal-01294409, doi:10.1007/s00454-016-9782-6.

[3] M. Ebbens. Systole of regular hyperbolic surfaces with an application to Delaunay triangulations. *9th International Conference on Curves and Surfaces*, 2018.

[4] I. Iordanov and M. Teillaud. Implementing Delaunay triangulations of the Bolza surface. In *Proceedings of the Thirty-third International Symposium on Computational Geometry*, pages 44:1–44:15, 2017. doi:10.4230/LIPIcs.SoCG.2017.44