Webs of rational curves on surfaces

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Lines play a central role in Euclidean geometry and are rational curves of minimal degree. Through any two points in the plane exists a unique line and the family of lines in the plane is 2-dimensional. A *simple family* is an algebraic family of minimal degree rational curves that covers a real surface $X \subset \mathbb{P}^n$, such that a general curve in this family is smooth outside the singular locus of X. Moreover, we assume that the dimension of a simple family is as large as possible. A *simple curve* is a curve that belongs to some simple family. The intersection product of two simple families that cover X is defined as the number of intersections between a general curve in the first family and a general curve in the second family, outside the singular locus of X. The *simple family graph* $\mathcal{G}(X)$ is defined as follows:

- Each vertex is a simple family of X. The vertex is labeled with the dimension of the simple family.
- We draw between two simple families an edge if their intersection product is at least two. We label the edge with this intersection product.

For example, $\mathcal{G}(\mathbb{P}^2)$ consists of a single vertex labeled 2. The graph $\mathcal{G}(\mathbb{S}^2)$ of the projective 2-sphere $\mathbb{S}^2 \subset \mathbb{P}^3$ consists of a single vertex with label 3.

Theorem. *If an edge in $\mathcal{G}(X)$ has label ≥ 5, then $\mathcal{G}(X)$ has at most 2160 vertices. If $\mathcal{G}(X)$ contains a vertex with label ≥ 3 then $X \cong \mathbb{S}^2$.*

Below are examples for $\mathcal{G}(X)$ in case simple curves are conics. We colored the vertices according to their corresponding simple curves. All vertices are labeled 1 and all edges are labeled 2.

ring torus | Blum cyclide | Blum cyclide | sextic del Pezzo surface

In the images, black areas on the surface are bordered by exactly three simple curves. Many hexagonal patterns emerge and therefore such families are also known as *hexagonal webs* [1]. Hexagonal webs of simple curves have been characterized on the plane [2], on the sphere [4, 5, 3] and on Darboux cyclides [6]. The following theorem is a characterization of hexagonal webs of simple curves on any algebraic surface.

Theorem. *If $X \subset \mathbb{P}^n$ is a surface, then simple families that correspond to three mutually disconnected vertices in the graph $\mathcal{G}(X)$ form a hexagonal web.*

References

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