Elastic Splines II: unicity of optimal s-curves and curvature continuity

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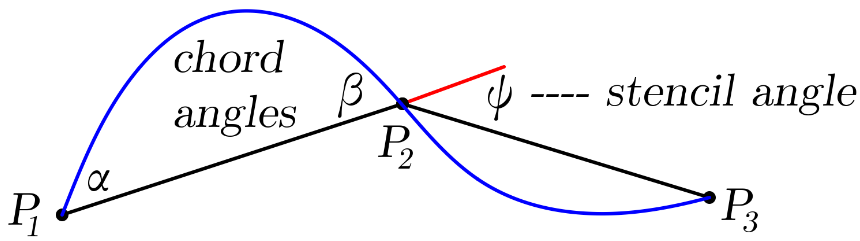
In this talk, we propose a new model for the draftsman's spline. Let $P_1, P_2, \ldots, P_n$ be given interpolation points in the plane. An **interpolating curve** is a smooth curve that passes sequentially through the interpolation points. The naive model states that the draftsman's spline takes the shape of an interpolating curve whose bending energy is minimal. It was observed at General Motors in 1965 (Birkhoff, Burchard, de Boor, Thomas) that no such curve exists because one can always describe a (very long and rounded) interpolating curve with arbitrarily small bending energy. Instead, they posited that the curve have a bending energy that is 'locally minimal' (i.e., minimal among all nearby interpolating curves). Such interpolating curves are called **nonlinear splines** and they reported that the pieces of a nonlinear spline are segments of rectangular elastica. Nonlinear splines do not always exist, but it is known (Lee & Forsythe, Brunett) that when they exist, they are curature continuous. However, the prickly issue of existence has never been adequately resolved, despite noteworthy efforts of Fisher, Jerome and Linnér.

Our approach to modeling the draftsman's spline is to seek a curve, with minimal bending energy, from a certain set of admissable interpolating curves. In [1], we introduced the notion of an s-curve (a curve that first turns monotonically at most 180° in one direction and then turns monotonically at most 180° in the opposite direction) and then declared that an interpolating curve is *admissible* if its pieces (connecting one interpolation point to the next) are s-curves. Existence of a curve (called an **elastic spline**) with minimal bending among all admissible interpolating curves is proved in [1]. Elastic splines were computed in an open source program *Curve Ensemble*, associated with [2], and it was observed that these curves were sometimes curvature continuous and sometimes not. It was also learned that an additional restriction is in order; namely, each piece of an admissable curve should be an s-curve with chord angles $\leq 90°$. With this resctricted notion of admissiblity, existence is still guaranteed and such curves with minimal bending energy are called **restricted elastic splines**. In our view, the question of existence for nonlinear splines has been transformed into the question of curvature continuity for restricted elastic splines.



The restriction that chord angles are $\leq 90°$ brings much needed 'law and order' to the subject and we will describe a strong analogy with natural cubic splines. The fruit of this analogy is a proof that if the chord angles of a restricted elastic spline are all $< 90°$, then it is curvature continuous. This naturally raises the challenge of finding sufficient conditions on the interpolation points that will ensure that chord angles are $< 90°$. Towards this end, we have identified a critical angle $\Psi \approx 37°$ such that if all of the stencil angles are $< \Psi$, then the restricted elastic spline is curvature continuous. We have also proved that Ψ is sharp.

Joint work with: Albert Borbély.

References

[1] A. Borbely & M.J. Johnson. Elastic splines I: existence. *Constr. Approx.*, **40**:189–218, 2014.

[2] M.J. Johnson & H.S. Johnson. A constructive framework for minimal energy planar curves. *Appl. Math. Comp.*, **276**:172–181, 2016.