Adaptive anisotropic approximation of multivariate functions by piecewise constants

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Let \( \Omega \subset \mathbb{R}^d \), \( d \geq 2 \), be a bounded domain. A finite collection \( \Delta \) of subdomains \( \omega \subset \Omega \) is called a partition of \( \Omega \) provided that \( \omega \cap \omega' = \emptyset \), for any \( \omega, \omega' \in \Delta, \omega \neq \omega' \), and \( \sum_{\omega \in \Delta} |\omega| = |\Omega| \), where \( |\cdot| \) is the Lebesgue measure. We call a partition \( \Delta \) convex if every cell \( \omega \in \Delta \) is convex, and for \( N \in \mathbb{N} \), denote by \( D_N \) the set of all convex partitions of \( \Omega \) comprising \( N \) cells.

For \( 1 \leq q \leq \infty \) and \( k \in \mathbb{N} \), by \( W^k(\Omega) \) we denote the standard Sobolev space of functions \( f : \Omega \to \mathbb{R} \) endowed with the semi-norm

\[
|f|_{W^k_q(\Omega)} = \sum_{\alpha \in \mathbb{Z}^d : |\alpha| = k} \|D^\alpha f\|_{L^q(\Omega)},
\]

where \( \alpha \in \mathbb{Z}^d \) is the multi-index.

For a partition \( \Delta \) of \( \Omega \), we denote by \( S(\Delta) \) the space of functions \( s : \Omega \to \mathbb{R} \) constant on every \( \omega \in \Delta \). For \( 1 \leq p \leq \infty \), we define the error of the best \( L_p \)-approximation of a function \( f : \Omega \to \mathbb{R} \) by piecewise constant functions on \( N \) cells:

\[
E_N(f)_p := \inf_{\Delta \in D_N} \inf_{s \in S(\Delta)} \|f - s\|_{L_p(\Omega)}.
\]

It was established in [1] that for \( f \in W^1_q(\Omega) \), the quantity \( E_N(f)_p \) behaves as \( O(N^{-1/d}) \) as \( N \to \infty \) provided that \( \frac{1}{d} + \frac{1}{p} - \frac{1}{q} > 0 \). O. Davydov in [2] constructed an approximation method with anisotropic partitions allowing to improve the estimate of the order of \( E_N(f)_p \) to \( O(N^{-2/(d+1)}) \) as \( N \to \infty \) for functions \( f \in W^2_p(\Omega) \). He also indicated that this \( \frac{2}{d+1} \) is the saturation order of piecewise constant approximation. In the current work we were able to estimate the order of \( E_N(f)_p \) for functions \( f \in W^2_q(\Omega) \) for a wide range of parameters \( p \) and \( q \), and show that such approximation order can be achieved by a sort of greedy algorithms.

**Theorem 1.** Let \( \Omega \subset \mathbb{R}^d \) be a bounded domain, \( 1 \leq p \leq \infty \) and \( 1 \leq q \leq \infty \) be such that \( \frac{2}{d+1} + \frac{1}{p} - \frac{1}{q} \geq 0 \), and let \( f \in W^2_q(\Omega) \). Then

\[
E_N(f)_p \leq C(d,p,q)N^{-\frac{2}{d+1}} \left(|f|_{W^2_q(\Omega)}^q + |f|_{W^2_q(\Omega)}^q\right)^{\frac{1}{q}},
\]

with the constant \( C(d,p,q) \) independent on \( f \).

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**References**
