Adaptive anisotropic approximation of multivariate functions by piecewise constants

Oleksandr Kozynenko
Oles Honchar Dnipro National University, Dnipro, Ukraine
kozinenkoalex@gmail.com

Let $\Omega \subset \mathbb{R}^d$, $d \geqslant 2$, be a bounded domain. A finite collection Δ of subdomains $\omega \subset \Omega$ is called a *partition* of Ω provided that $\omega \cap \omega' = \varnothing$, for any $\omega, \omega' \in \Delta$, $\omega \neq \omega'$, and $\sum_{\omega\in\Delta} |\omega| = |\Omega|$, where $|\cdot|$ is the Lebesgue measure. We call a partition Δ *convex* if every cell $\omega \in \Delta$ is convex, and for $N \in \mathbb{N}$, denote by $\mathfrak{D}_N$ the set of all convex partitions of Ω comprising N cells.

For $1 \leqslant q \leqslant \infty$ and $k \in \mathbb{N}$, by $W_q^k(\Omega)$ we denote the standard Sobolev space of functions $f : \Omega \to \mathbb{R}$ endowed with the semi-norm

$$|f|_{W_q^k(\Omega)} = \sum_{\alpha\in\mathbb{Z}_+^d\,:\,|\alpha|=k} \|D^\alpha f\|_{L_q(\Omega)}\,,$$

where $\alpha \in \mathbb{Z}_+^d$ is the multi-index.

For a partition Δ of Ω, we denote by $\mathcal{S}(\Delta)$ the space of functions $s : \Omega \to \mathbb{R}$ constant on every $\omega \in \Delta$. For $1 \leqslant p \leqslant \infty$, we define the error of the best L_p-approximation of a function $f : \Omega \to \mathbb{R}$ by picewise constant functions on N cells:

$$E_N(f)_p := \inf_{\Delta\in\mathfrak{D}_N} \inf_{s\in S(\Delta)} \|f - s\|_{L_p(\Omega)}\,.$$

It was established in [1] that for $f \in W_q^1(\Omega)$, the quantity $E_N(f)_p$ behaves as $O\left(N^{-1/d}\right)$ as $N \to \infty$ provided that $\frac{1}{d}+\frac{1}{p}-\frac{1}{q} > 0$. O. Davydov in [2] constructed an approximation method with anisotropic partitions allowing to improve the estimate of the order of $E_N(f)_p$ to $O\left(N^{-2/(d+1)}\right)$ as $N \to \infty$ for functions $f \in W_p^2(\Omega)$. He also indicated that this $\frac{2}{d+1}$ is the saturation order of piecewise constant approximation. In the current work we were able to estimate the order of $E_N(f)_p$ for functions $f \in W_q^2(\Omega)$ for a wide range of parameters p and q, and show that such approximation order can be achieved by a sort of greedy algorithms.

Theorem 1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $1 \leqslant p \leqslant \infty$ and $1 \leqslant q \leqslant \infty$ be such that $\frac{2}{d+1}+\frac{1}{p}-\frac{1}{q} \geqslant 0$, and let $f \in W_q^2(\Omega)$. Then*

$$E_N(f)_p \leqslant C(d,p,q) N^{-\frac{2}{d+1}} \left(|f|^q_{W_q^1(\Omega)} + |f|^q_{W_q^2(\Omega)}\right)^{\frac{1}{q}},$$

with the constant $C(d,p,q)$ independent on f.

Joint work with: Prof. O.Davydov, Prof. D.Skorokhodov.

References

[1] Birman M. S.,Solomyak M. Z. Piecewise polynomial approximation of functions of the classes W_p^α(in Russian). *Matematicheskii sbornik*, 1967, 73 (115), no. 3, 331-355. English translation in *Math. USSR-Sb.*, 1967, 2, no. 3, 295-317.

[2] Davydov O. Approximation by piecewise constants on convex partitions. *J. Approx. Theory*, 164 (2012), 346-352.