Approximation by Bernstein polynomials with integer coefficients

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Approximation of continuous functions by algebraic polynomials with integer coefficients is a fascinating subject. To address it, besides the usual tools of Analysis, we also need results from Algebra and Number Theory.

Not every continuous function f(x) can be approximated in the uniform norm by algebraic polynomials with integer coefficients on a given compact interval [a, b] on the real line. Clearly, we must have that f(x) acquires integer values on every integer in [a, b]. There are, however, less obvious, even subtle, necessary conditions. They concern relations between function values at specific points, or the length of the interval.

On the other hand, as Bernstein [1] and Kantorovich [3] showed, on closed intervals of length less than 1, which contain no integer, the best approximation in the uniform norm by algebraic polynomials with integer coefficients is no substantially different than the real-coefficient case. The situation on [0, 1] is similar.

Following L. Kantorovich, we will consider the Bernstein polynomials with integer coefficients given by

$$\widetilde{B}_n(f,x) := \sum_{k=0}^n \left[f\left(\frac{k}{n}\right) \binom{n}{k} \right] x^k (1-x)^{n-k},$$

defined for $f \in C[0,1]$ and $x \in [0,1]$. Above $[\alpha]$ denotes the largest integer that is less than or equal to the real α . As is known, $\widetilde{B}_n(f)$ approximates f in the uniform norm on [0,1] iff f(0) and f(1) are integers.

We will present necessary and sufficient conditions for simultaneous approximation by B_n and its modification, in whose definition the integer part is replaced with the nearest integer. This modification actually has better approximation properties, subject to weaker restrictions. It is noteworthy that for the derivatives of order two and higher, the necessary conditions for both operators are stricter than for the first derivative.

Also, we will give direct and weak converse estimates of the rate of simultaneous approximation for both operators. They are similar to and make use of the results established in [2].

Finally, we will make several observations concerning shape preservation by these operators.

References

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