# Syzygies and Implicitization for Translational Surfaces 

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A translational surface is a rational tensor product surface generated from two rational space curves by translating either one of these curves parallel to itself in such a way that each of its points describes a curve that is a translation along the other curve. Translational surfaces [1], ruled surface, swept surfaces, surface of revolution, along with low degree surfaces such as quadratic surfaces [2], Steiner surfaces [3], cubic surfaces, and cyclides are basic modeling surfaces that are widely used in computer aided geometric design and geometric modeling.

Since translational surfaces are generated from two space curves, translational surfaces have simple representations. The simplest and perhaps the most common representation of a translational surface is given by the rational parametric representation $\mathbf{h}^{*}(s ; t)=\mathbf{f}^{*}(s)+\mathbf{g}^{*}(t)$, where $\mathbf{f}^{*}(s)$ and $\mathbf{g}^{*}(t)$ are two rational space curves. Translational surfaces represented by $\mathbf{h}^{*}(s ; t)=\mathbf{f}^{*}(s)+\mathbf{g}^{*}(t)$ have been investigated by differential geometers, and also studied from a geometric modeling point of view.

Translational surfaces defined by $\mathbf{h}^{*}(s ; t)=\mathbf{f}^{*}(s)+\mathbf{g}^{*}(t)$ are not translation invariant: translating both curves $\mathbf{f}^{*}$ and $\mathbf{g}^{*}$ by the vector $\mathbf{v}$ translates the surface $\mathbf{h}^{*}$ by the vector $2 \mathbf{v}$. One would like to define translational surfaces in such a way that translating the two generating curves by the same vector $\mathbf{v}$, also translates every point on the surface by the vector $\mathbf{v}$. In this presentation, we offer an alternative definition of translational surfaces given by the rational parametric representation $\mathbf{h}^{*}(s ; t)=\frac{\mathbf{f}^{*}(s)+\mathbf{g}^{*}(t)}{2}$, where $\mathbf{f}^{*}(s)$ and $\mathbf{g}^{*}(t)$ are two rational space curves. Under this definition, these translational surfaces consist of all the midpoints of all the lines joining a point on $\mathbf{f}^{*}$ to a point on $\mathbf{g}^{*}$, so these translational surfaces are invariant under rigid motions: translating and rotating the two generating curves translates and rotates these translational surfaces by the same amount. Hence, applying a rigid motion to a translational surface can be achieved by applying the same rigid motion to the two rational space curves that generate the surface. Therefore, one can control these translational surfaces simply by manipulating the generating curves.

In this presentation, we will investigate the translational surfaces given by the rational parametric representation $\mathbf{h}^{*}(s ; t)=\frac{\mathbf{f}^{*}(s)+\mathbf{g}^{*}(t)}{2}$. Our main goal is to utilize syzygies to study translational surfaces. We will construct three special syzygies for a translational surface from the $\mu$-basis of one of the generating space curves. In addition, we will examine many properties of translational surfaces, and compute the implicit equation and singularities from these three special syzygies.

The outline of the presentation is structured as the following. First, we introduce the definition of translational surfaces, provide a few examples of translational surfaces generated from two rational space curves, and establish a few properties of translational surfaces. Second, we recall the notions of syzygies and $\mu$-bases for rational curves and surfaces, then we study syzygies of translational surfaces, relate the syzygies of the generating curves to the syzygies of the corresponding translational surface. Third, we construct three special syzygies for a translational surface from a $\mu$-basis of one of the generating space curves, and show how to use these three special syzygies to compute the implicit equation of a translational surface. Finally, we observe that the techniques used in this paper can be applied with only minor modifications to the translational surfaces defined by $\mathbf{h}^{*}(s ; t)=a \mathbf{f}^{*}(s)+b \mathbf{g}^{*}(t)$, where $a, b$ are real numbers and $a b \neq 0$. In the case of $a=b=1$, we provide a necessary and sufficient condition for a rational tensor product surface to be a translational surface.

Joint work with: Ron Goldman.

## References

[1] S. Pérez-Díaz, and L. Shen, Parametrization of translational surfaces, Proceeding SNC' 14 Proceedings of the 2014 Symposium on Symbolic-Numeric Computation, 128-129.
[2] R. Goldman, Quadrics of Revolution, IEEE Computer Graphics and Applications, Vol. 3 (1983), 68-76.
[3] X. Wang and F. Chen, Implicitization, parameterization and singularity computation of Steiner surfaces using moving surfaces, Journal of Symbolic Computation, 47(2012), 733-750.

